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Cowling–Price theorem and characterization of heat kernel on symmetric spaces

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Abstract. We extend the uncertainty principle, the Cowling–Price theorem, on non-compact Riemannian symmetric spaces X. We establish a characterization of the heat kernel of the Laplace–Beltrami operator on X from integral estimates of the Cowling–Price type.

Keywords. Hardy's theorem; spherical harmonics; symmetric space; Jacobi function; heat kernel.

1. Introduction

Our starting point in this paper is the classical Hardy's theorem [14]: if a measurable function f on $\mathbb{R}$ satisfies $|f(x)| \leq C\mathrm{e}^{-ax^2}$, $|\hat{f}(x)| \leq C\mathrm{e}^{-bx^2}$, $x \in \mathbb{R}$ for positive a and b with $ab > \frac{1}{4}$, then $f = 0$ almost everywhere. But if $a \cdot b = \frac{1}{4}$ then f is a constant multiple of the Gauss kernel $\mathrm{e}^{-a|x|^2}$. The first assertion of the theorem, thanks to a number of articles in the recent past (e.g. [8,10,28,30]), may now be viewed as instances of a fairly general phenomenon in harmonic analysis of Lie groups known as Hardy's uncertainty principle. At the same time, several variants of the above decay conditions have been employed in the studies by many mathematicians, under which the first assertion has been proved. Notable among them are the integral estimates on f and $\widehat{f}$ introduced by Cowling and Price [7]. Due to intrinsic difficulties, however, research remains incomplete in most of these cases as to the second assertion of the theorem, that is, which are the functions that satisfy the sharpest possible decay conditions and it is this aspect that we take up for study in this paper.

In this article we consider the problem on a Riemannian symmetric space X of non-compact type. We realize X as G/K, where G is a connected non-compact semisimple Lie group with finite center and K is a maximal compact subgroup of G. Thus a function on X is a right K-invariant function on G and $\widehat{f}$ is the operator Fourier transform on the space of representations of the group G. On these spaces Narayanan and Ray have discovered [24] that the correct Hardy-like estimates involve the heat kernel of the Laplace–Beltrami operator on X. In another paper they have also proved [25] the first assertion, i.e., the uncertainty theorem under the integral estimates of Cowling and Price (C–P). As we take up here the case of sharpest decay in the sense of C–P, our results reinforce their findings, functions satisfying sharp C–P estimates indeed involve the heat kernel. There is however a further element of intricacy, as we arrive at a hierarchy of sharp decay conditions,

instead of just one sharpest decay. These are obtained by looking at the case $a \cdot b = \frac{1}{4}$ and tempering the exponentials with suitable polynomials in the estimates. Our main result in this direction is Theorem 4.1.

We go on to notice that since the matrix coefficients of the principal series representations are explicitly known in the case of rank 1 symmetric spaces, we can exploit the relation between the group-Fourier transform and Jacobi transform on $\mathbb{R}$. This allows us to vary the polynomial in the C–P estimates which now characterize not only the bi-invariant heat kernel but also some allied class of left K-finite functions on X arising again as the solutions of the heat equation of the Laplace–Beltrami operator.

As is often the case with analysis of the non-compact symmetric spaces, there is a strong analogy with the Euclidean space $\mathbb{R}^n$ looked upon as the homogeneous space $M(n)/O(n)$, where $M(n)$ is the Euclidean motion group and $O(n)$ is the orthogonal group. Here the Laplace operator Δ_n generates all $M(n)$-invariant differential operators on $\mathbb{R}^n$ and the plane waves $\mathrm{e}^{-i\lambda x\omega}$ can be thought of as the basic eigenfunctions of Δ_n. We consider the Fourier transform of a function f in polar coordinates: $\widehat{f}(\lambda,\omega) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{-i\lambda x\cdot\omega} f(x)\mathrm{d}x$ for $\lambda \in \mathbb{R}^+, \omega \in S^{n-1}$ and further expand $\widehat{f}(\lambda,\cdot)$ in terms of the eigenfunctions of the Laplace operator of S^{n-1} – we call the coefficients of this expansion as Fourier coefficients of $\widehat{f}(\lambda,\cdot)$. Then we can replace the integral estimate on $\widehat{f}$ by similar estimates on the Fourier coefficients of the function $\widehat{f}(\lambda,\cdot)$ on S^{n-1} and obtain a different version of the C–P type result for $\mathbb{R}^n$ characterizing the heat kernel of the Laplace operator Δ_n (Theorem 3.3).

Back on the symmetric space X, we recall that the symmetric spaces $X = G/K$ of rank 1 are also non-compact two-point homogeneous spaces (see [16]) like $\mathbb{R}^n$. We can go over to the analogue of the polar coordinates through the KAK decomposition of G. It follows that the space K/M is $S^{m_\gamma+m_{2\gamma}}$ where M is the centralizer of A in K and $m_\gamma, m_{2\gamma}$ are the multiplicities of the two positive roots γ and 2γ respectively. We impose conditions on the Fourier coefficients of the Helgason Fourier transform $\tilde{f}(\lambda,\cdot)$ treating them as functions on K/M and get a version of the C–P result which corresponds to that of the Euclidean case mentioned above. We prefer to put our results for the rank 1 case in this direction instead of reworking the results of the general case. Our result in this case can be thought of as vindicating Helgason's programme initiated in his Paley–Wiener theorem (see [17]).

Hardy's theorem on semisimple Lie groups was first taken up by Sitaram and Sundari [30]. It inspired many articles in recent times including ours (see [12] for a survey of these results). After this work was finished, we came to know about a recent paper by Bonami *et al* [6]. They have obtained a Cowling–Price type result on $\mathbb{R}^n$, as a corollary of a more general theorem of Beurling which they prove for $\mathbb{R}^n$ in this paper. Our result on $\mathbb{R}^n$ is only slightly different from what they obtain in [6]. However note that Beurling's theorem has not yet been proved for symmetric spaces. We may also mention here that the last part of our work is influenced by a recent paper by Thangavelu [33].

Throughout this article for a $p \in [1,\infty]$, p' denotes its conjugate, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. For two functions f_1 and f_2, $f_1(x) \asymp f_2(x)$ means there exists two positive constants C, C' such that $Cf_2(x) \leq f_1(x) \leq C' f_2(x)$. We follow the practice of using C, C' etc. to denote constants (real or complex) whose value may change from one line to the next. We use subscripts and superscripts of C when needed to indicate their dependence on parameters. Required preliminaries are given at the beginning of the sections.

This paper has some overlap with a paper of Andersen [1]. (In fact the author is kind enough to refer to the pre-print version of this paper.)

2. A complex analytic result

In this section we shall prove a result of complex analysis which will be useful throughout this paper.

Lemma 2.1. *Let* $\mathscr{Q} = \{\rho e^{i\psi} | \rho > 0, \psi \in (0, \frac{\pi}{2})\}$. *Suppose a function g is analytic on* $\mathscr{Q}$ *and continuous on the closure* $\overline{\mathscr{Q}}$ *of* $\mathscr{Q}$. *Also suppose that for* $p \in [1, \infty)$ *and for constants* $C > 0, a > 0$ *and* $m \geq 0$, (i) $|g(x+iy)| \leq C e^{ax^2}(1+|z|)^m$ *for* $z = x+iy \in \overline{\mathscr{Q}}$ *and* (ii) $\int_0^\infty |g(x)|^p \mathrm{d}x < \infty$. *Then for* $\psi \in [0, \frac{\pi}{2}]$ *and* $\sigma \in \mathbb{R}^+$, $\int_\sigma^{\sigma+1} |g(\rho e^{i\psi})| \mathrm{d}\rho \leq C \max\{e^a, (\sigma+1)^{1/p}\}(\sigma+2)^{2m}$.

Proof. For $m = 0$ this is proved in [7]. For $m > 0$, we define $h(z) = \frac{g(z)}{(i+z)^{2m}}$ for $\operatorname{Im} z > 0$. Then the lemma follows by applying the case $m = 0$ to the function h.

Lemma 2.2. *Let g be an entire function on* $\mathbb{C}^d$ *such that,*

(i) $|g(z)| \leq C e^{a|\operatorname{Re} z|^2}(1+|\operatorname{Im} z|)^m$ *for some* $m > 0, a > 0$,

(ii) $\int_{\mathbb{R}^d} \frac{|g(x)|^q}{(1+|x|)^s} |Q(x)| \mathrm{d}x < \infty$, *for some* $q \geq 1, s > 1$ *and a polynomial Q of degree M in d variables.*

Then g is a polynomial. Moreover $\deg g \leq \min\{m, \frac{s-M-d}{q}\}$ *and if* $s \leq q+M+d$, *then g is a constant.*

Proof. Once we prove that g is a polynomial, it would be clear from (i) and (ii) respectively that, $\deg g \leq m$ and $\deg g < \frac{s-M-d}{q}$ and hence if $s < q+M+d$, then g is a constant.

We will first assume that $d = 1$.

Since for a scalar $\alpha, g - \alpha$ also satisfies the above conditions, we may and will assume that $g(0) = 0$. Consider the function h given by $g(z) = zh(z)$. Thus h is an entire function which satisfies (i). Define

$$H(z) = \frac{h(z)}{(1-iz)^{s/q}} \quad \text{for } z \in \mathscr{Q},$$

where $\mathscr{Q} = \{\rho e^{i\psi} | \rho > 0, \psi \in (0, \frac{\pi}{2})\}$. Then H is analytic on $\mathscr{Q}$ and continuous on $\overline{\mathscr{Q}}$. Since $|1-iz|^{s/q} \geq 1$ for $z \in \overline{\mathscr{Q}}$, we have

$$|H(z)| \leq C e^{a(\operatorname{Re} z)^2}(1+|\operatorname{Im}\, z|)^m \quad \text{for } z \in \overline{\mathscr{Q}}.$$

Now

$$\int_0^\infty |H(x)|^q \mathrm{d}x = \int_0^\infty \frac{|h(x)|^q}{|1-ix|^s} \mathrm{d}x$$
$$= \int_0^\infty \frac{|h(x)|^q}{(1+x^2)^{s/2}} \mathrm{d}x < \infty,$$

by continuity of the integrand and (ii). Thus H satisfies the conditions of Lemma 2.1 and hence

$$\int_\sigma^{\sigma+1} |H(\rho e^{i\psi})| \mathrm{d}\rho \leq A \max\{e^a, (\sigma+1)^{1/q}\}(\sigma+2)^{2m}.$$

Since $|1-i\rho e^{i\psi}| \le (1+\rho)$, it follows that

$$\int_{\sigma}^{\sigma+1} |h(\rho e^{i\psi})| d\rho \le A \max\{e^{a}, (\sigma+1)^{1/q}\}(\sigma+2)^{2m}(\sigma+2)^{s/q}.$$

Considering the functions $H_1(z) = \overline{h(\overline{z})}/(1-iz)^{s/q}$, $H_2(z) = \overline{h(-\overline{z})}/(1-iz)^{s/q}$, $H_3(z) = h(-z)/(1-iz)^{s/q}$ for $z \in \mathscr{Q}$ we get that for large σ

$$\int_{\sigma}^{\sigma+1} |h(\rho e^{i\psi})| d\rho \le A'(\sigma+2)^{2m}(\sigma+2)^{(s+1)/q}, \text{ for all } \psi. \tag{2.1}$$

By Cauchy's integral formula

$$|h^{r}(0)| \le r!(2\pi)^{-1}\int_{0}^{2\pi} |h(\rho e^{i\psi})|\rho^{-r} d\psi.$$

For large σ integrating both sides with respect to ρ in $[\sigma, \sigma+1]$, we get

$$\begin{aligned} |h^{r}(0)| &\le r!(2\pi)^{-1}\sigma^{-r}\int_{\sigma}^{\sigma+1}\int_{0}^{2\pi} |h(\rho e^{i\psi})| d\psi\, d\rho \\ &\le Cr!(2\pi)^{-1}\sigma^{-r}(\sigma+2)^{(s+1)/q+2m} \end{aligned}$$

by (2.1). Let $\sigma \to \infty$, then $h^{r}(0) = 0$ for all r after some stage. Thus h is a polynomial and hence so is g.

The proof of the lemma now proceeds by induction. Assuming the statement of the lemma to be true for $d = 1, 2, \dots, n-1$, we prove it for $d = n$. To this end we write $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ as $z = (\tilde{z}, z_n)$ where $\tilde{z} = (z_1, z_2, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$. We notice that for $x \in \mathbb{R}^n, (1+|x|) \le (1+|\tilde{x}|)(1+|x_n|)$. Thus the hypothesis of the lemma implies $\int_{\mathbb{R}^n} \frac{|g(\tilde{x}, x_n)|^q}{(1+|\tilde{x}|)^s(1+|x_n|)^s} |Q(\tilde{x}, x_n)| d\tilde{x}\, dx_n < \infty$.

By Fubini's theorem, we have a subset $B_1 \subset \mathbb{R}^{n-1}$ of full measure such that for $\tilde{x} \in B_1, \int_{\mathbb{R}} \frac{|g(\tilde{x}, x_n)|^q}{(1+|x_n|)^s} |Q(\tilde{x}, x_n)| dx_n < \infty$. Further, since Q is a non-zero polynomial we may choose B_1 so that for each $\tilde{x} \in B_1$, $Q(\tilde{x}, \cdot)$ is a non-zero polynomial of one variable. Thus if $\tilde{x} \in B_1$, the entire function $g_{\tilde{x}}$ of one variable $g_{\tilde{x}} = g(\tilde{x}, z)$ satisfies the condition (i) and (ii) of the lemma and consequently, is a polynomial of one variable. Writing $g(\tilde{z}, z_n) = \sum_{m=0}^{\infty} a_m(\tilde{z}) z_n^m, \tilde{z} \in \mathbb{C}^{n-1}, z_n \in \mathbb{C}$, where each a_m is entire function of $\tilde{z}$, what we have just proved means that for $\tilde{x} \in B_1, a_m(\tilde{x})$ is zero except for finitely many values of m (depending on $\tilde{x}$). But each a_m being an entire function, if a_m is not identically zero, $a_m(\tilde{x}) = 0$ only if $\tilde{x} \in N_m$ where $N_m \subset \mathbb{R}^{n-1}$ has measure zero. If for infinitely many m say for $m \in \{m_1 < m_2 < \cdots < m_i < \cdots\}, a_m \not\equiv 0$, we would have $B_1 \subset \cup_{i=1}^{\infty} N_{m_i}$, a contradiction. This shows that $g(\tilde{z}, z_n) = \sum_{m=0}^{M} a_m(\tilde{z}) z_n^m$ for some positive integer M. A similar argument with the role of $\tilde{z}$ and z_n reversed and the induction hypothesis for $d = n-1$ will give an upper bound on the degree of the monomials in $\tilde{z} = (z_1, z_2, \dots, z_{n-1})$ occurring in g. Together, it is proved that $g(\tilde{z}, z_n)$ is a polynomial separately in $\tilde{z}$ and z_n and hence a polynomial in $z = (\tilde{z}, z_n) \in \mathbb{C}^n$ (see [26]). The degree of g is estimated at the beginning of the proof. Thus the proof is complete.

3. Euclidean spaces

For $f \in L^1(\mathbb{R}^n)$ let

$$\widehat{f}(y) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(x) e^{-ix\cdot y} dx, \tag{3.1}$$

where by $x \cdot y$ we mean the inner product of x and y in $\mathbb{R}^n$. Let $p_t(x) = (4\pi t)^{-n/2} \mathrm{e}^{-|x|^2/4t}$, $t > 0$ be the heat kernel associated with the Laplacian on $\mathbb{R}^n$. Then $\widehat{p_t}(y) = (2\pi)^{-n/2} \mathrm{e}^{-t|y|^2}$.

We can have the Cowling–Price theorem [7] extended as follows:

Theorem 3.1. *Let f be a measurable function on $\mathbb{R}^n$ which satisfies, for $p, q \in [1, \infty)$ and $s, t > 0$*

(i) $\int_{\mathbb{R}^n} \frac{|f(x)|^p \mathrm{e}^{\frac{p}{4s}|x|^2}}{(1+|x|)^k} \mathrm{d}x < \infty$ *and* (ii) $\int_{\mathbb{R}^n} \frac{|\widehat{f}(\lambda)|^q \mathrm{e}^{qt|\lambda|^2}}{(1+|\lambda|)^l} \mathrm{d}\lambda < \infty$ *for any $k \in (n, n+p]$ and $l \in (n, n+q]$. Then*

(a) *if $s < t$ then $f \equiv 0$,*
(b) *if $s = t$ then f is a constant multiple of p_t,*
(c) *if $s > t$ then there exists infinitely many linearly independent functions satisfying* (i) *and* (ii).

We do not give our proof of the theorem as it runs along the same lines as the proof, later in this paper, of the corresponding theorem on Riemannian symmetric spaces (Theorem 4.1). Instead we proceed to a variant of the Cowling–Price theorem where the usual Fourier transform is replaced by spherical harmonic coefficients of $\widehat{f}(\lambda, \cdot)$, where $\widehat{f}(\lambda, \omega) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-i\lambda x \cdot \omega} f(x) \mathrm{d}x, \lambda \in \mathbb{R}^+, \omega \in S^{n-1}$.

For a non-negative integer m, let $\mathscr{H}_m$ denote the space of spherical harmonics of degree m on S^{n-1}. For a fixed $S_m \in \mathscr{H}_m$ and $f \in L^1(\mathbb{R}^n), n \geq 2$, the Fourier coefficients of f in the angular variable are defined by

$$f_m(|x|) = f_m(r) = \int_{S^{n-1}} f(rx') S_m(x') \mathrm{d}x' \quad \text{for almost every } r > 0. \tag{3.2}$$

In the Fourier domain we define

$$F_m(\lambda) = \lambda^{-m} \int_{S^{n-1}} \widehat{f}(\lambda, \omega) S_m(\omega) \mathrm{d}\omega, \quad \lambda > 0. \tag{3.3}$$

We also note that if $F_m = 0$ for all $S_m \in \mathscr{H}_m$ for all $m \in \mathbb{N}$, then by the uniqueness of the Fourier transform $f_m \equiv 0$ for all $S_m \in \mathscr{H}_m$, $m \in \mathbb{N}$ and so $f \equiv 0$.

Lemma 3.2. *Let $f \in L^1(\mathbb{R}^n)$ and for a fixed $S \in \mathscr{H}_m, F_m$ and f_m are as defined above. Then*

$$F_m(\lambda) = C \int_{\mathbb{R}^{n+2m}} f_m(|x|) |x|^{-m} \mathrm{e}^{-i\lambda x \cdot \omega} \mathrm{d}x \quad \textit{for any } \omega \in S^{n+2m-1}, \lambda > 0. \tag{3.4}$$

Proof. We recall two results. If f is radial on $\mathbb{R}^n$, $f(x) = f_0(|x|)$, then $\widehat{f}$ is also radial and we have for $y \in \mathbb{R}^n, |y| = r > 0$:

$$\widehat{f}(y) = \widehat{f}(|y|) = \widehat{f}(r) = r^{-((n-2)/2)} \int_0^\infty f_0(s) J_{\frac{n-2}{2}}(rs) s^{n/2} \mathrm{d}s, \tag{3.5}$$

where $J_k, k \geq \frac{1}{2}$ is the Bessel function (see [32], pp. 154–155). We also have ([16], p. 25 Lemma 3.6),

$$\int_{S^{n-1}} \mathrm{e}^{i\lambda r x' \omega} S_m(\omega) \mathrm{d}\omega = C_{n,m} \frac{J_{\frac{n}{2}+m-1}(\lambda r)}{(\lambda r)^{\frac{n}{2}-1}} S_m(x'). \tag{3.6}$$

From (3.3) and the definition of the Fourier transform in polar coordinates we have,

$$F_m(\lambda) = (2\pi)^{-n/2}\lambda^{-m}\int_{\mathbb{R}^n}\left(\int_{S^{n-1}} \mathrm{e}^{i\lambda x\cdot\omega} S_m(\omega)\mathrm{d}\omega\right) f(x)\mathrm{d}x.$$

Let $x = rx'$ where $x' \in S^{n-1}$ and $|x| = r$. By (3.6), (3.2) and (3.5) we get,

$$\begin{aligned} F_m(\lambda) & \\ &= C_{n,m}(2\pi)^{-n/2}\lambda^{-m}\int_0^\infty\int_{S^{n-1}} \frac{J_{\frac{n}{2}+m-1}(\lambda r)}{(\lambda r)^{\frac{n}{2}-1}} r^{n-1} S_m(x') f(rx')\mathrm{d}x'\mathrm{d}r \\ &= C_{n,m}(2\pi)^{-n/2}\lambda^{-m}\int_0^\infty f_m(r)\frac{J_{\frac{n}{2}+m-1}(\lambda r)}{(\lambda r)^{\frac{n}{2}-1}} r^{n-1}\mathrm{d}r \\ &= C_{n,m}(2\pi)^{-n/2}\lambda^{-(n+2m-2)/2}\int_0^\infty f_m(r) r^{-m} J_{\frac{n}{2}+m-1}(\lambda r) r^{\frac{n+2m}{2}}\mathrm{d}r \\ &= C_{n,m}(2\pi)^{-n/2}\int_{\mathbb{R}^{n+2m}} f_m(|x|)|x|^{-m}\mathrm{e}^{-i\lambda\omega\cdot x}\mathrm{d}x. \end{aligned} \tag{3.7}$$

This establishes the lemma.

Theorem 3.3. *Let $f\colon \mathbb{R}^n \longrightarrow \mathbb{C}$ be a measurable function such that for $p,q \in [1,\infty), s,t > 0$ and for each non-negative integer m and $S_m \in \mathscr{H}_m$,*

(i) $\int_{\mathbb{R}^n} \frac{|f(x)|^p \mathrm{e}^{\frac{p}{4s}|x|^2}}{(1+|x|)^k}\mathrm{d}x < \infty$ *and* (ii) $\int_{\mathbb{R}^+}\frac{|F_m(\lambda)|^q \mathrm{e}^{qt\lambda^2}}{(1+\lambda)^l}\mathrm{d}\lambda < \infty$ *for any $k \in (n, n+p]$ and $l \in (1, 1+q]$. Then*

(a) *if $s < t$ then $f \equiv 0$,*
(b) *if $s = t$ then f is a constant multiple of p_t,*
(c) *if $s > t$ then there exist infinitely many linearly independent functions satisfying* (i) *and* (ii).

Proof. It follows from (i) that $f \in L^1(\mathbb{R}^n)$ and let f_m be defined as above with respect to $S \in \mathscr{H}_m$.

Let $I = \int_0^\infty |f_m(r)|^p \frac{\mathrm{e}^{\frac{p}{4s}r^2}}{(1+r)^k} r^{n-1}\mathrm{d}r$. Then by Holder's inequality and (i) we get

$$I \le C_m \int_{\mathbb{R}^n} \frac{|f(x)|^p \mathrm{e}^{\frac{p}{4s}|x|^2}}{(1+|x|)^k}\mathrm{d}x < \infty. \tag{3.8}$$

Using polar coordinates in (3.4) and taking $\lambda = u+iv \in \mathbb{C}$ we get

$$\begin{aligned}
&|F_m(u+iv)|\\
&\quad \le \int_0^\infty \int_{S^{n+2m-1}} |f_m(r)| r^{n+2m-1} r^{-m} \mathrm{e}^{|v|r} \mathrm{d}x'\,\mathrm{d}r \ (\text{as } |x'| = |\omega| = 1)\\
&\quad = C\int_0^\infty \frac{|f_m(r)| \mathrm{e}^{\frac{1}{4s}r^2} r^{\frac{n-1}{p}}}{(1+r)^{k/p}} (1+r)^{k/p}\, r^{\frac{1-n}{p}} r^{(n+m-1)} \mathrm{e}^{-\frac{1}{4s}r^2+|v|r}\mathrm{d}r\\
&\quad \le C\left[\int_0^\infty \mathrm{e}^{(-\frac{1}{4s}r^2+|v|r)p'} (1+r)^{kp'/p}\, r^{(\frac{1-n}{p})p'} r^{(n+m-1)p'} \mathrm{d}r\right]^{1/p'}\\
&\qquad \text{(by Holder's inequality and (3.8))}\\
&\quad = C\left[\int_0^\infty \mathrm{e}^{(-\frac{p'}{4s}(r-2s|v|))^2} (1+r)^{kp'/p}\, r^{n-1+mp'} \mathrm{d}r\right]^{1/p'} \mathrm{e}^{sv^2}\\
&\quad \le C\mathrm{e}^{sv^2}(1+s|v|)^R \text{ for some non-negative } R.
\end{aligned}$$

If $s \le t$ then $|F_m(u+iv)| \le C\mathrm{e}^{tv^2}(1+t|v|)^R$. From the existence of the integral (3.4) for all $u+iv \in \mathbb{C}$ estimate one can easily show that F_m is an entire function. Let $G(z) = F_m(z)\mathrm{e}^{tz^2}$. Then, $|G(z)| \le c\mathrm{e}^{t(\mathrm{Re}\, z)^2}(1+t|\mathrm{Im} z|)^R$ and $\int_{\mathbb{R}} \frac{|G(x)|^q}{(1+|x|)^l}\mathrm{d}x < \infty$ by (ii).

Applying Lemma 2.2 for $n=1$ on $G(z)$ we have $F_m(\lambda) = C_m \mathrm{e}^{-t\lambda^2} P_m(\lambda)$ for $\lambda \in \mathbb{R}$, where P is a polynomial whose degree depends on R and l. Since $1 < l \le 1+q$, we see from (ii) that the polynomial is constant and hence $F_m(\lambda) = C_m \mathrm{e}^{-t\lambda^2}$. Therefore $f_m(|x|) = C_m|x|^m p_t(x)$. But if $s<t$, it follows that the integral I does not exist unless $C_m = 0$. Since the argument applies to all $S_m \in \mathscr{H}_m, m = 0,1,\ldots, f \equiv 0$.

If $s = t$, then the integral in (i) is again infinite unless $C_m = 0$ whenever $m > 0$. Thus $f(x) = f_0(x) = C_0 p_t(x)$ by the uniqueness of Fourier transform.

When $s > t$, let $s > t_0 > t$ for some t_0. Let $H(x)$ be a solid harmonic of degree $k \ge 1$, i.e. $H(r,\omega) = r^k S(w), r \in \mathbb{R}^+, \omega \in S^{n-1}$, for some $S \in \mathscr{H}_k$. Then it is easy to see that $H \cdot p_{t_0}$ satisfies conditions (i) and (ii) of the theorem. This completes the proof.

4. Symmetric spaces

Let G be a connected non-compact semisimple Lie group with finite center and let K be a fixed maximal compact subgroup of G. Our set-up for the rest of the paper is the Riemannian symmetric space $X = G/K$ equipped with the intrinsic metric d. Whenever convenient, we will treat a function f on X also as a right K-invariant function on G.

Let $G = KAN$ (resp. $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$) be an Iwasawa decomposition of G (resp. $\mathfrak{g}$) where $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}$ are the Lie algebras of G, K, A and N respectively. Let $\mathfrak{a}^*$ and $\mathfrak{a}^*_{\mathbb{C}}$ respectively be the real dual of $\mathfrak{a}$ and its complexification. Let $P = MAN$ be the minimal parabolic subgroup corresponding to this Iwasawa decomposition where M is the centralizer of A in K. Let M' be the normalizer of A in K. Then $W = M'/M$ is the (restricted) Weyl group of (G,A) which acts on $\mathfrak{a}_{\mathbb{C}}$ and on its dual $\mathfrak{a}^*_{\mathbb{C}}$. Let $\Sigma(\mathfrak{g},\mathfrak{a})$ be the set of restricted roots, $\Sigma^+ \subset \Sigma(\mathfrak{g},\mathfrak{a})$ be the set of positive restricted roots which is chosen once for all and $\Sigma_0^+ \subset \Sigma^+$ be the set of indivisible positive roots. Let us denote the underlying set of

simple roots by Δ_0 and the corresponding positive Weyl chamber in $\mathfrak{a}$ by $\mathfrak{a}^+$. Then G has the Cartan decomposition $G = K\overline{A^+}K$, where $A^+ = \exp \mathfrak{a}^+$. Let $\langle\,,\rangle$ be the Killing form. Suppose the real rank of G is n, i.e. $\dim \mathfrak{a} = n$ and $\{H_1, H_2, \ldots, H_n\}$ an orthonormal basis (with respect to $\langle\,,\rangle$) of $\mathfrak{a}$. For $\lambda \in \mathfrak{a}^*$, let $H_\lambda \in \mathfrak{a}$ correspond to λ via $\langle\,,\rangle$, i.e. $\lambda(H) = \langle H_\lambda, H\rangle$ and let $\lambda_j = \langle H_\lambda, H_j\rangle$. Then $\lambda = (\lambda_1, \ldots, \lambda_n)$ identifies $\mathfrak{a}^*$ with $\mathbb{R}^n$ and hence $\mathfrak{a}^*_{\mathbb{C}}$ with $\mathbb{C}^n$.

Among the representations of G, those relevant for analysis on X are the so-called (minimal) spherical principal series representations $\pi_\lambda, \lambda \in \mathfrak{a}^*_{\mathbb{C}} = \mathbb{C}^n$. The representation π_λ is induced from the representation $\xi_0 \otimes \exp(\lambda) \otimes 1$ of $P = MAN$ (where ξ_0 and 1 are the trivial representations of M and A respectively and $\lambda \in \mathbb{C}^n$ acts as a character of the vector subgroup A) and is unitary only if $\lambda \in \mathfrak{a}^* = \mathbb{R}^n$. For all $\lambda \in \mathbb{C}^n$, we can realize π_λ on the Hilbert space $L^2(K/M)$ (compact picture) where for $x \in G$,

$$\begin{aligned}(\pi_\lambda(x)u)(k) &= \mathrm{e}^{-(i\lambda+\rho_0)(H(x^{-1}k))} u(K(x^{-1}k)) \\ &= \mathrm{e}^{(i\lambda+\rho_0)(A(xK,kM))} u(K(x^{-1}k)), \quad k \in K, u \in L^2(K/M),\end{aligned}$$

where $\rho_0 = \frac{1}{2}\Sigma_{\gamma\in\Sigma^+} m_\gamma \gamma, m_\gamma$ being the multiplicity of the root γ.

Here for $g \in G, K(g) \in K, H(g) \in \mathfrak{a}$ and $N(g) \in N$ are the *parts* of g in the Iwasawa decomposition $G = KAN$, i.e. $g = K(g) \exp H(g) N(g)$. The vector-valued inner product $A(\cdot,\cdot)$ is defined as $A(gK, kM) = -H(g^{-1}k)$ (see [17]). We can choose an orthonormal basis of $L^2(K/M)$ consisting of K-finite vectors. Let u be an arbitrary element of this basis and let e_0 be the K-fixed vector in it. From the action of $\pi_\lambda(x)$ defined above, it is clear that for all $x \in G, \langle e_0, \pi_{i\rho_0}(x)e_0\rangle = 1$ and when $u \neq \mathrm{e}_0, \langle u, \pi_{i\rho_0}(x)e_0\rangle = 0$. For a suitable function f on X, let $\widehat{f}(\lambda) = \int_{G/K} f(x)\pi_\lambda(x)\mathrm{d}x$ denote its Fourier transform with respect to π_λ. The (u, e_0)th matrix coefficient of the operator $\widehat{f}(\lambda)$, denoted by $\widehat{f_u}(\lambda)$ is given by

$$\widehat{f_u}(\lambda) = \int_X f(x)\langle u, \pi_\lambda(x)e_0\rangle \mathrm{d}x.$$

Let dk and da respectively be the Haar measures on K and A and $\int_K \mathrm{d}k = 1$. Let d be the distance on X induced by the Riemannian metric on it. We define $\sigma(x) = \mathrm{d}(xK, o)$, where $o = eK, x \in X$. Then $\sigma(\exp H) = |H| = \langle H, H\rangle^{1/2}$ for all $H \in \mathfrak{a}$. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n$, by $|\lambda|$ we shall mean its usual norm while $|\lambda|_{\mathbb{R}}$ will stand for $\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2$.

The following estimate of Harish-Chandra ([15], §9) will be useful for us:

$$\mathrm{e}^{-\rho_0(H)} \leq \Xi(\exp H) \leq C\mathrm{e}^{-\rho_0(H)}(1+|H|)^{|\Sigma_0^+|} \quad \text{for all } H \in \overline{\mathfrak{a}_0^+}, \tag{4.1}$$

where $\Xi(x)$ is ϕ_0, i.e. the elementary spherical function with parameter 0 and $|\Sigma_0^+|$ is the cardinality of Σ_0^+.

For $a \in A$, by $\log a$ we shall mean an element in $\mathfrak{a}$ such that $\exp \log a = a$. The Haar measure dx on G can be normalized so that $\mathrm{d}x = J(a)\mathrm{d}k_1\,\mathrm{d}a\,\mathrm{d}k_2$, where $J(a) = \Pi_{\gamma\in\Sigma^+}(\mathrm{e}^{\gamma(\log a)} - \mathrm{e}^{-\gamma(\log a)})^{m_\gamma}$ is the Jacobian of the Cartan decomposition of G. Clearly,

$$|J(a)| \leq C\mathrm{e}^{2\rho_0(\log a)}. \tag{4.2}$$

For notational convenience we will from now on use $\lambda_{\mathbb{R}}$ and λ_I respectively for the real and the imaginary parts of $\lambda \in \mathbb{C}$ (instead of the usual $\mathrm{Re}\,\lambda$ and $\mathrm{Im}\,\lambda$).

Let $\delta \in \widehat{K}$ and let d_δ be the degree of δ. Let us assume that $u \in L^2(K/M)$ transforms according to δ. Then, from the well-known estimate of the elementary spherical function ϕ_λ (see e.g. [13], Prop. 4.6.1) and using the arguments of Miličić ([23], p. 83) (see also [28], 4.2) we have

$$|\langle u, \pi_\lambda(a)e_0\rangle| \le (d_\delta)^{1/2}\phi_{\lambda_I}(x) \le C_\delta \mathrm{e}^{\lambda_I^+(\log a)}\Xi(a) \quad \text{for } a \in A^+ \text{ and } \lambda \in \mathfrak{a}^*_{\mathbb{C}}, \tag{4.3}$$

where λ_I^+ is the Weyl translate of λ_I which is dominant, i.e. belongs to the positive Weyl chamber.

We need the following estimate for the Plancherel measure $\mu(\lambda) = |c(\lambda)|^{-2}$ ($c(\lambda)$ being the Harish-Chandra's c-function) (see [2], p. 394):

$$|c(\lambda)|^{-2} \asymp \Pi_{\gamma\in\Sigma_0^+}\langle\lambda,\gamma\rangle^2(1+|\langle\lambda,\gamma\rangle|)^{m_\gamma+m_{2\gamma}-2} \quad \text{for } \lambda \in \mathfrak{a}^*. \tag{4.4}$$

Using the identification of $\mathfrak{a}^*_{\mathbb{C}}$ and $\mathfrak{a}^*$ with $\mathbb{C}^n$ and $\mathbb{R}^n$ respectively, we will write an element λ in $\mathbb{C}^n$ (resp. $\mathbb{R}^n$) as $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $\lambda_i \in \mathbb{C}$ (resp. $\in \mathbb{R}$) for $i = 1, \ldots, n$ and V for the degree of the polynomial

$$\Pi_{\gamma\in\Sigma_0^+}\langle\lambda,\gamma\rangle^2(1+\langle\lambda,\gamma\rangle)^{m_\gamma+m_{2\gamma}-2}. \tag{4.5}$$

With this preparation we come to the main theorem of this section.

Theorem 4.1. *Let $f \in L^1(X) \cap L^2(X)$ and let $p, q \in [1,\infty)$. Suppose for some $k, l, a, b \in (0,\infty)$,*

$$\int_X \frac{|f(x)\Xi(x)^{\frac{2}{p}-1}\mathrm{e}^{a\sigma(x)^2}|^p}{(1+\sigma(x))^k}\mathrm{d}x < \infty \tag{4.6}$$

and

$$\int_{\mathbb{R}^n} \frac{\|\widehat{f}(\lambda)\|_2^q \mathrm{e}^{qb(\lambda_1^2+\lambda_2^2+\cdots+\lambda_n^2)}}{(1+|\lambda|)^l}\mu(\lambda)\mathrm{d}\lambda < \infty, \tag{4.7}$$

where $\widehat{f}(\lambda) = \int_X f(x)\pi_\lambda(x)\mathrm{d}x$ is the operator values of Fourier transform of f at the spherical principal series π_λ and $\|\widehat{f}(\lambda)\|_2$ is the Hilbert–Schimdt norm of $\widehat{f}(\lambda)$.

(i) *If $a \cdot b = \frac{1}{4}$, then $\widehat{f}_u(\lambda) = P_{u,b}(\lambda)\mathrm{e}^{-b(\lambda_1^2+\cdots+\lambda_n^2)}, \lambda \in \mathbb{C}^n$, for some polynomial $P_{u,b}$ with $\deg P_{u,b} < \min\left\{\frac{2|\Sigma_0^+|}{p'} + \frac{k}{p} + 1, \frac{l-V-n}{q}\right\}$, where V is given by (4.5).*
If also $l \le q + V + n$, then $P_{u,b}$ is a constant.

(ii) *If $a \cdot b > \frac{1}{4}$, then $f \equiv 0$.*

Proof. Let us recall that $\widehat{f}_u(\lambda) = \int_G f(x)\langle u, \pi_\lambda(x)e_0\rangle\mathrm{d}x$ is the $(u,0)$th matrix coefficient of $\widehat{f}(\lambda)$ for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, if the integral exists.

We shall show that $\widehat{f}_u(\lambda)$ exists and is an entire function in $\lambda \in \mathbb{C}^n$ and

$$|\widehat{f}_u(\lambda)| \le C\mathrm{e}^{b|\lambda_I|^2}(1+|\lambda_I|)^{k'} \quad \text{for all } \lambda \in \mathbb{C}^n, \text{ for some } k' > 0. \tag{4.8}$$

We rewrite the condition (4.6) as

$$\int_{A^+} \frac{|f(h)\Xi(h)^{\frac{2}{p}-1}\mathrm{e}^{a|\log h|^2}|^p}{(1+|\log h|)^k} J(h)\mathrm{d}h < \infty \quad \text{for all } h \in A^+. \tag{4.9}$$

Then using (4.3), for all $\lambda \in \mathbb{C}^n$, we have

$$\begin{aligned}
|\widehat{f}_u(\lambda)| &= \left| \int_{A^+} f(h)\langle u, \pi_\lambda(h)e_0\rangle J(h)\mathrm{d}h \right| \\
&\le C \int_{A^+} \left| \frac{f(h)\Xi(h)^{\frac{2}{p}-1}\mathrm{e}^{a|\log h|^2}}{(1+|\log h|)^{k/p}} \right| \mathrm{e}^{-a|\log h|^2} \\
&\quad \times (1+|\log h|)^{k/p}\mathrm{e}^{\lambda_I^+(\log h)}\Xi(h)^{2(1-\frac{1}{p})}J(h)\mathrm{d}h.
\end{aligned}$$

From (4.9) and applying (4.1) and (4.2), we get for $k' = \frac{2|\Sigma_0^+|}{p'} + \frac{k}{p}$,

$$\begin{aligned}
|\widehat{f}_u(\lambda)| &\le C \cdot \left(\int_{A^+} \mathrm{e}^{-p'a|\log h|^2}\mathrm{e}^{p'\lambda_I^+(\log h)}\mathrm{e}^{-2\rho_0(\log h)} \right. \\
&\quad \left. \times (1+|\log h|)^{k'p'}\mathrm{e}^{2\rho_0(\log h)}\mathrm{d}h \right)^{1/p'} \\
&= C \cdot \left(\int_{A^+} \mathrm{e}^{-p'a|\log h|^2}\mathrm{e}^{p'\lambda_I^+(\log h)}(1+|\log h|)^{k'p'}\mathrm{d}h \right)^{1/p'} \\
&\le C \cdot \left(\int_{\mathfrak{a}} \mathrm{e}^{-p'a|H|^2}\mathrm{e}^{p'\lambda_I^+(H)} \cdot (1+|H|)^{k'p'}\mathrm{d}H \right)^{1/p'},
\end{aligned}$$

where $H = \log h$ and $\mathrm{d}H$ is the Lebesgue measure on $\mathfrak{a}$. We recall that $H_{\lambda_I^+}$ corresponds to λ_I^+ as mentioned above so that $|\lambda_I^+| = |H_{\lambda_I^+}|$. Then,

$$\begin{aligned}
&|\widehat{f}_u(\lambda)| \\
&\quad \le C \cdot \mathrm{e}^{\frac{1}{4a}|H_{\lambda_I^+}|^2} \left(\int_{\mathfrak{a}} \mathrm{e}^{-p'a\langle H-\frac{1}{2a}H_{\lambda_I^+}, H-\frac{1}{2a}H_{\lambda_I^+}\rangle} \cdot (1+|H|)^{k'p'}\mathrm{d}H \right)^{1/p'}.
\end{aligned}$$

From this using translation invariance of Lebesgue measure, for $\lambda \in \mathfrak{a}_{\mathbb{C}}^* = \mathbb{C}^n$,

$$\begin{aligned}
|\widehat{f}_u(\lambda)| &\le C \cdot \mathrm{e}^{\frac{1}{4a}|H_{\lambda_I^+}|^2}(1+|H_{\lambda_I^+}|)^{k'} \left(\int_{\mathfrak{a}} \mathrm{e}^{-p'a|H|^2}(1+|H|)^{k'p'}\mathrm{d}H \right)^{1/p'} \\
&\le C \cdot (1+|\lambda_I|)^{k'}\mathrm{e}^{\frac{1}{4a}|\lambda_I|^2} \int_{\mathfrak{a}} \mathrm{e}^{-a|H|^2}(1+|H|)^{k'}\mathrm{d}H \quad \text{as } |\lambda_I^+| = |\lambda_I| \\
&= C' \cdot (1+|\lambda_I|)^{k'}\mathrm{e}^{b|\lambda_I|^2} \quad \left(\text{as } b = \frac{1}{4a} \right).
\end{aligned}$$

The analyticity of $\widehat{f}_u(\lambda)$ is a result of an usual argument from Cauchy's integral formula and Fubini's theorem.

Let $g(\lambda) = e^{b(\lambda_1^2+\lambda_2^2+\cdots+\lambda_n^2)}\widehat{f}_u(\lambda)$. Then $|g(\lambda)| \le C' e^{b|\lambda_{\mathbb{R}}|^2}(1+|\lambda_I|)^{k'}$. Also from condition (4.7), $\int_{\mathbb{R}^n} \frac{|g(\lambda)|^q}{(1+|\lambda|)^s}\mu(\lambda)\mathrm{d}\lambda < \infty$.

It follows from (4.4) that $|c(\lambda)|^{-2}$ can be replaced by a polynomial. Hence by Lemma 2.2, g is a polynomial say $P_{u,b}(\lambda)$ and $\deg P_{u,b} \le k', \deg P_{u,b} < \frac{l-V-n}{q}$. Thus, $\widehat{f}_u(\lambda) = P_{u,b}(\lambda)\cdot e^{-b(\lambda_1^2+\lambda_2^2+\cdots+\lambda_n^2)}$, for all $\lambda \in \mathbb{C}^n$.

If $l \le q+V+n$, then clearly $P_{u,b}$ is a constant. This proves (i).

If $a\cdot b > \frac{1}{4}$, then we can choose positive constants a_1, b_1 such that $a > a_1 = \frac{1}{4b_1} > \frac{1}{4b}$. Then f and $\widehat{f}$ also satisfy (4.6) and (4.7) with a and b replaced by a_1 and b_1 respectively. Therefore $\widehat{f}_u(\lambda) = P_{u,b_1}(\lambda)e^{-b_1(\lambda_1^2+\cdots+\lambda_n^2)}$. But then $\widehat{f}_u$ cannot satisfy (4.7) for any u unless $P_{u,b_1} \equiv 0$, which implies $f \equiv 0$.

The following related results come as immediate consequences of Theorem 4.1.

COROLLARY 4.2. [25]

Let f be a measurable function on X and let $p,q \in [1,\infty)$. Suppose for $a,b \in \mathbb{R}^+$ with $a\cdot b \ge \frac{1}{4}$,

$$\text{(i)} \qquad \int_X |f(x)\Xi(x)^{\frac{2}{p}-1}e^{a\sigma(x)^2}|^p \mathrm{d}x < \infty, x \in X$$

and

$$\text{(ii)} \qquad \int_{\mathbb{R}^n} \|\widehat{f}(\lambda)\|^q e^{qb(\lambda_1^2+\lambda_2^2+\cdots+\lambda_n^2)}\mu(\lambda)\mathrm{d}\lambda < \infty, \lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n) \in \mathbb{R}^n$$

then $f \equiv 0$.

Proof. If $a\cdot b = \frac{1}{4}$, then clearly f and $\widehat{f}$ satisfy the conditions of Theorem 4.1 with $l < q+V+n$ and hence $\widehat{f}_u(\lambda) = Ce^{-\frac{1}{4a}(\lambda_1^2+\cdots+\lambda_n^2)}$. From (ii) it follows that $C = 0$.

The proof for the case $a\cdot b > \frac{1}{4}$, proceeds as in Theorem 4.1(ii).

COROLLARY 4.3. [24,29]

Let f be a measurable function on X. Suppose for positive constants a, b and C and for $r \ge 0$ with $a\cdot b = \frac{1}{4}$,

$$\text{(i)} \qquad |f(x)| \le C\Xi(x)e^{-a\sigma(x)^2}(1+\sigma(x))^r, x \in X$$

and

$$\text{(ii)} \qquad \|\widehat{f}(\lambda)\| \le C' e^{-b(\lambda_1^2+\lambda_2^2+\cdots+\lambda_n^2)}, \lambda \in \mathbb{R}^n$$

then $\widehat{f}_0(\lambda) = Ce^{-\frac{1}{4a}(\lambda_1^2+\cdots+\lambda_n^2)}$ for $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n) \in \mathbb{C}^n$ and $\widehat{f}_u \equiv 0$ for any non-trivial u.

Proof. From (i) and (ii) we observe that f and $\widehat{f}$ satisfy the estimates of Theorem 4.1 with $l \le q+V+n$. So $\widehat{f}_u(\lambda) = C_u e^{-\frac{1}{4a}(\lambda_1^2+\cdots+\lambda_n^2)}$ for any $\lambda \in \mathbb{C}^n$. But since for any non-trivial u, the matrix coefficient function $\Phi^{u,0}_{i\rho}(x) = \langle \pi_{i\rho_0}(x)u, e_0\rangle$ of $\pi_{i\rho_0}$ is identically zero, it follows that $\widehat{f}_u(\lambda) \equiv 0$. The only non-zero part is $\widehat{f}_0(\lambda) = C_0 e^{-\frac{1}{4a}(\lambda_1^2+\cdots+\lambda_n^2)}$ for $\lambda \in \mathbb{C}^n$.

COROLLARY 4.4. [30]

Let f be a measurable function on X. Suppose for positive constants a,b and C with $a\cdot b>\frac{1}{4}$,

(i) $|f(x)|\le C\mathrm{e}^{-a\sigma(x)^2}, x\in X$

(ii) $\|\widehat{f}(\lambda)\|\le C'\mathrm{e}^{-b(\lambda_1^2+\lambda_2^2+\cdots+\lambda_n^2)}, \lambda\in\mathbb{R}^n$

then $f\equiv 0$.

Proof. As $a>\frac{1}{4b}$ we can choose positive constants a_1 and b_1 such that $a>a_1=\frac{1}{4b_1}>\frac{1}{4b}$. Thus f and $\widehat{f}$ satisfy (i) and (ii) of the previous corollary with a and b replaced by a_1 and b_1 respectively. Therefore it follows that $\widehat{f_0}(\lambda)=C_1\mathrm{e}^{-b_1(\lambda_1^2+\cdots+\lambda_n^2)}$ and $\widehat{f_u}\equiv 0$ for any non-trivial u. But as $b>b_1$, $\widehat{f_0}$ as above cannot satisfy (ii) unless $C_1=0$.

Sharpness of the estimates. To complete the picture we should consider the case $a\cdot b<\frac{1}{4}$ and also show the optimality of the factor $\Xi^{\frac{2}{p}-1}$ considered in Theorem 4.1. We provide an example below to show that if we substitute $\Xi^{\frac{2}{p}-\ell}$ for $\Xi^{\frac{2}{p}-1}$, with $\ell\in[0,1)$ in (4.6), then there are infinitely many linearly independent functions, satisfying this modified estimate while their Fourier transforms still satisfy (4.7) in Theorem 4.1. Also we will see that the same example will show that if $a\cdot b<\frac{1}{4}$ in the hypothesis of Theorem 4.1, then there are infinitely many linearly independent functions satisfying (4.6) and (4.7).

Example. Let $G=SL_2(\mathbb{C})$ and K be its maximal compact subgroup $SU(2)$ and $X=SL_2(\mathbb{C})/SU(2)$. Then,

$$A=\left\{a_t=\begin{pmatrix}\mathrm{e}^t & 0\\ 0 & \mathrm{e}^{-t}\end{pmatrix}\,\middle|\, t\in\mathbb{R}\right\}.$$

Let α be the unique element in Σ^+ given by $\alpha(\log a_t)=2t$ which occurs with multiplicity 2. Then $\sigma(a_t)=2|t|$. Every $\lambda\in\mathbb{C}$ can be identified with an element in $\mathfrak{a}_{\mathbb{C}}^*$ by $\lambda=\lambda\alpha$. In this identification, $\rho=2$, the unitary spherical principal series representations are given by elements in $\mathbb{R}$, the Plancherel measure $|c(\lambda)|^{-2}=|\lambda|^2$ and the elementary spherical function $\phi_\lambda(a_t)=\sin(2\lambda t)/\lambda\sinh(2t)$ (see [16], p. 432).

For a suitable function f on $\mathbb{R}$, let $\tilde{f}$ be its Euclidean Fourier transform and $C_c^\infty(\mathbb{R})_{\text{even}}$ be the set of even functions in $C_c^\infty(\mathbb{R})$.

We define a bi-invariant function g on G by prescribing its spherical Fourier transform $\widehat{g}(\lambda)=\tilde{\psi}(2\lambda)\widehat{h}(\lambda)P(2\lambda)$ for $\lambda\in\mathbb{R}$, where $\psi\in C_c^\infty(\mathbb{R})_{\text{even}}$ with support $[-\zeta,\zeta]$ for some $\zeta>0$, $\widehat{h}(\lambda)=\mathrm{e}^{-\lambda^2/4}$ for $\lambda\in\mathbb{R}$ and P is an even polynomial on $\mathbb{R}$. It follows from the characterization of the bi-invariant functions in the Schwartz space $S(G)$ (see [13]) that $g\in S(G)$, g is bi-invariant and hence can be thought of as a function on X.

Therefore Fourier inversion gives us

$$\begin{aligned}g(a_t)&=C\cdot\int_{\mathbb{R}}\tilde{\psi}(2\lambda)\mathrm{e}^{-\lambda^2/4}P(2\lambda)\frac{\sin(2\lambda t)}{\lambda\sinh(2t)}\lambda^2\mathrm{d}\lambda\\ &=\frac{C}{\sinh(2t)}\int_{\mathbb{R}}\tilde{\psi}(\lambda)\cdot\mathrm{e}^{-\lambda^2/16}\lambda P(\lambda)\sin(\lambda t)\mathrm{d}\lambda\\ &=\frac{C}{\sinh(2t)}(\psi_1*_E h)(t),\end{aligned}$$

where $h(t) = \mathrm{e}^{-4t^2}, \psi_1 \in C_c^\infty(\mathbb{R})$ is the odd function supported on $[-\zeta, \zeta]$ such that $\tilde{\psi}_1(\lambda) = \lambda P(\lambda)\tilde{\psi}(\lambda)$ (i.e. ψ_1 is certain derivative of ψ) and $*_E$ the Euclidean convolution. Therefore for large t and hence, choosing C sufficiently large, for all $t \in \mathbb{R}$,

$$\begin{aligned} |g(a_t)| &\leq \frac{C}{|\sinh 2t|} \cdot \mathrm{e}^{-4t^2} \mathrm{e}^{8\zeta t} \\ &\leq C \cdot \mathrm{e}^{-\sigma(a_t)^2} \mathrm{e}^{-(1-4\zeta)2t} \\ &= C \cdot \mathrm{e}^{-\sigma(a_t)^2} \mathrm{e}^{-(1-4\zeta)2t} \\ &\leq C \cdot \mathrm{e}^{-\sigma(a_t)^2} \Xi(a_t)^{(1-4\zeta)}. \end{aligned}$$

Now if we choose ζ so that $\ell = (1-4\zeta) > 0$, then for all $x \in X, g$ satisfies

$$|g(x)| \leq C\mathrm{e}^{-\sigma(x)^2} \Xi(x)^\ell (1+\sigma(x))^M \quad \text{for } \ell \in (0,1) \text{ and for some } M > 0. \tag{4.10}$$

Its Fourier transform is $\widehat{g}(\lambda) = \tilde{\psi}(\lambda)\widehat{h}(\lambda)P(\lambda), \lambda \in \mathbb{R}$. As $\tilde{\psi}$ is bounded on $\mathbb{R}$,

$$|\widehat{g}(\lambda)| \leq C' \mathrm{e}^{-|\lambda|^2/4} (1+|\lambda|)^N \quad \text{on } \mathbb{R} \text{ for some } N > 0. \tag{4.11}$$

By (4.10), (4.11), (4.1) and (4.2) this g clearly satisfies for $p, q \in [1, \infty)$ and $\ell \in (0,1)$,

$$\int_X \frac{|g(x)\Xi(x)^{\frac{2}{p}-\ell} \mathrm{e}^{a\sigma(x)^2}|^p}{(1+\sigma(x))^k} \mathrm{d}x < \infty \tag{4.12}$$

and

$$\int_{\mathbb{R}} \frac{|\widehat{g}(\lambda)|^q \mathrm{e}^{qb\lambda^2}}{(1+|\lambda|)^l} \mu(\lambda) \mathrm{d}\lambda < \infty, \tag{4.13}$$

where $a = 1, b = \frac{1}{4}$ and $k > 3 + Mp, l > 3 + Nq$.

Since we can choose any $\psi \in C_c^\infty(\mathbb{R})_{\text{even}}$ and any even polynomial $P(\lambda)$ to construct such a function g, we have infinitely many linearly independent functions which satisfy (4.12) and (4.13).

Case $a \cdot b < \frac{1}{4}$. Notice that from (4.10) and (4.11) it follows that there exist constants $C_1, C_2 > 0$ for which the function g constructed above satisfies the estimates

$$|g(x)| \leq C_1 \mathrm{e}^{-\frac{1}{2}\sigma(x)^2} \Xi(x) \tag{4.14}$$

and

$$|\widehat{g}(\lambda)| \leq C_2 \mathrm{e}^{-\frac{1}{5}|\lambda|^2} \quad \text{on } \mathbb{R}. \tag{4.15}$$

Therefore

$$\int_X \frac{|g(x)\Xi(x)^{\frac{2}{p}-1} \mathrm{e}^{a'\sigma(x)^2}|^p}{(1+\sigma(x))^k} \mathrm{d}x < \infty \tag{4.16}$$

and

$$\int_{\mathbb{R}^n} \frac{|\widehat{g}(\lambda)|^q e^{qb'\lambda^2}}{(1+|\lambda|)^l}\mu(\lambda)\mathrm{d}\lambda < \infty, \tag{4.17}$$

where $a' = \frac{1}{2}$ and $b' = \frac{1}{5}$ and hence $a' \cdot b' < \frac{1}{4}$. In [27] we have used this example for $p = q = \infty$ case.

Characterization of the heat kernel. The heat kernel on X is an analogue of the Gauss kernel p_t on $\mathbb{R}^n$ (see §3). Let Δ be the Laplace–Beltrami operator of X. Then (see [31], Chapter v), $T_t = e^{t\Delta}, t > 0$ defines a semigroup (heat-diffusion semigroup) of operators such that for any $\phi \in C_c^\infty(X)$, $T_t\phi$ is a solution of $\Delta u = \partial u/\partial t$ and $T_t\phi \longrightarrow \phi$ a.e. as $t \longrightarrow 0$. For every $t > 0, T_t$ is an integral operator with kernel h_t, i.e. for any $\phi \in C_c^\infty(X), T_t\phi = \phi * h_t$. Then $h_t, t > 0$ are bi-invariant functions and h as a function of the variables $t \in \mathbb{R}^+$ and $x \in G/K$ is in $C^\infty(G \times \mathbb{R}^+)$ satisfying the properties:

(i) $\{h_t : t > 0\}$ form a semigroup under convolution $*$. That is, $h_t * h_s = h_{t+s}$ for $t, s > 0$.
(ii) $h_t(x)$ is a fundamental solution of $\Delta u = \partial u/\partial t$.
(iii) $h_t \in L^1(G) \cap L^\infty(G)$ for every $t > 0$.
(iv) $\int_X h_t(x)\mathrm{d}x = 1$ for every $t > 0$.

Thus we see that the heat kernel h_t on X retains all the nice properties of the classical heat kernel. It is well-known that h_t is given by (see [3]):

$$h_t(x) = \frac{1}{|W|}\int_{\mathfrak{a}^*} e^{-t(|\lambda|_{\mathbb{R}}^2 + |\rho_0|_{\mathbb{R}}^2)}\phi_\lambda(x)\mu(\lambda)\mathrm{d}\lambda, \tag{4.18}$$

where for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n, |z|_{\mathbb{R}} = z_1^2 + \cdots + z_n^2$, as defined earlier. That is, the spherical Fourier transform of $h_t, \widehat{h_t}(\lambda)_{0,0} = e^{-t(|\lambda|_{\mathbb{R}}^2 + |\rho_0|_{\mathbb{R}}^2)}$. It has been proved in [3] (Theorem 3.1(i)) that for any $t_0 > 0$, there exists $C > 0$ such that

$$h_t(\exp H) \le C t^{-n/2} e^{-|\rho_0|_{\mathbb{R}}^2 t - \langle \rho_0, H\rangle - \frac{|H|^2}{4t}}(1+|H|^2)^{\frac{d_X - n}{2}} \tag{4.19}$$

for $t_0 \ge t > 0$ and $H \in \overline{\mathfrak{a}^+}$, where $d_X = \dim X$.

Now the following elegant characterization of the heat kernel follows from Theorem 4.1 but we present it in the form of a theorem to stress the point.

Theorem 4.5. *Let f be a measurable function on X such that for some $t > 0$ and for $p, q \in [1, \infty)$,*

$$\int_X \frac{|f(x)\Xi(x)^{\frac{2}{p}-1} e^{\frac{1}{4t}\sigma(x)^2}|^p}{(1+\sigma(x))^k}\mathrm{d}x < \infty \tag{4.20}$$

and

$$\int_{\mathbb{R}^n} \frac{|\widehat{f}(\lambda) e^{t|\lambda|^2}|^q}{(1+|\lambda|)^l}\mu(\lambda)\mathrm{d}\lambda < \infty, \tag{4.21}$$

where $k > (d_X - n)p + 2|\Sigma_0^+| + n, V + n < l \le q + V + n$ (V is as in (4.5)) and $\int_X f(x)\mathrm{d}x = 1$. Then $f = h_t$.

Proof. Let $a = 1/4t$ and $b = t$. Then it follows from Theorem 4.1 that $\widehat{f_u}(\lambda) = C_u \cdot \mathrm{e}^{-t(\lambda_1^2+\lambda_2^2+\cdots+\lambda_n^2)} = C_u \cdot \mathrm{e}^{-t|\lambda|^2}$, because the condition $l \leq q+V+n$ forces the polynomial $P_{u,b}$ of Theorem 4.1 to be a constant. But when $u \neq e_0$, then as noted in the beginning of this section where we have described the representations, that $\langle u, \pi_{i\rho_0}(x)e_0\rangle = 0$ for all x and hence $\widehat{f_u} \equiv 0$. This implies that for any non-trivial $\delta \in \widehat{K}_0$, $f_\delta \equiv 0$, where $f_\delta = d_\delta \chi_\delta *_K f$ is the δth projection of f. Thus f is bi-invariant and its spherical Fourier transform $\widehat{f}(\lambda)_{0,0} = C_0 \cdot \mathrm{e}^{-t|\lambda|^2_{\mathbb{R}}}$. Again since, $\phi_{\pm i\rho_0}(\cdot) \equiv 1, \int_X f(x)\mathrm{d}x = \int_X f(x)\phi_{\pm i\rho_0}(x)\mathrm{d}x = \widehat{f}(\pm i\rho_0)_{0,0}$. Therefore the given initial condition reduces to $\widehat{f}(\pm i\rho_0)_{0,0} = 1$. From this, we have $\widehat{f}(\lambda)_{0,0} = \mathrm{e}^{-t(|\lambda|^2_{\mathbb{R}}+|\rho_0|^2_{\mathbb{R}})}$. This completes the proof in view of (4.18) above. It is clear from the estimates (4.1) and (4.19) that k and l taken above are good enough to accommodate h_t.

5. Symmetric spaces of rank 1

We shall revisit the Cowling–Price theorem from the point of view of §2 and relate it with the result obtained in the previous section. We shall play around with the polynomials in the denominator of the integrand of the C–P estimates. We will see that a larger class of solutions of the heat equation can in fact be characterized using these estimates, the usual heat kernel being one of them. The main technical tool in this section will be the Jacobi functions. We shall crucially use their relations with the Eisenstein integrals. Here the Jacobi functions will take the role played by the Bessel functions in §2.

Throughout this section the symmetric space X is of rank 1. We will continue to use the set-up and notation of the previous section, adapting to this particular case. Here $\rho_0 = \frac{1}{2}(m_\gamma + 2m_{2\gamma})$. For $\lambda \in \mathbb{C}$, the function $x \longrightarrow \mathrm{e}^{(i\lambda+\rho_0)A(x,b)}$ is a common eigenfunction of invariant differential operators on X. This motivates one to define Helgason–Fourier transform of a function as a generalization of the Fourier transform in polar coordinates:

$$\tilde{f}(\lambda, b) = \int_X f(x)\mathrm{e}^{(-i\lambda+\rho_0)A(x,b)}\mathrm{d}x, \tag{5.1}$$

where dx is the G-invariant measure on X. Let $\widehat{K}_0$ be the set of equivalence class of irreducible unitary representations of K which are class 1 with respect to M, i.e. contains an M-fixed vector – it is also known that an M-fixed vector is unique up to a multiple (see [22]). Let $(\delta, V_\delta) \in \widehat{K}_0, \delta$ different from the identity representation. Suppose $\{v_i | i = 1, \ldots, d_\delta\}$ is an orthonormal basis of V_δ of which v_1 is the M-fixed vector. Let $Y_{\delta,j}(kM) = \langle v_j, \delta(k)v_1\rangle, 1 \leq j \leq d_\delta$ and let Y_0 be the K-fixed vector, which we have denoted by e_0 in the previous section. Recall that $L^2(K/M)$ is the carrier space of the spherical principal series representations π_λ in the compact picture and $\{Y_{\delta,j}\colon 1 \leq j \leq d_\delta, \delta \in \widehat{K}_0\}$ is an ortho- normal basis for $L^2(K/M)$ adapted to the decomposition $L^2(K/M) = \Sigma_{\delta \in \widehat{K}_0} V_\delta$ (see [17]). As the space K/M is $S^{m_\gamma+m_2\gamma} = S^{2\alpha+1}$, this decomposition can be viewed as the spherical harmonic decomposition and therefore $Y_{\delta,j}$'s can be considered as the *spherical harmonics*.

For $\delta \in \widehat{K}_0, 1 \leq j \leq d_\delta, \lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $x \in X$, define,

$$\Phi^j_{\lambda,\delta}(x) = \int_K \mathrm{e}^{(i\lambda+\rho_0)A(x,kM)} Y_{\delta,j}(kM)\mathrm{d}k. \tag{5.2}$$

Then $\Phi^j_{\lambda,\delta}(x)$ is a matrix coefficient of the generalized spherical function (Eisenstein integral) $\int_K \mathrm{e}^{(i\lambda+\rho_0)A(x,kM)}\delta(k)\mathrm{d}k$ (see [17]). Again, $\Phi^j_{\lambda,\delta}(x) = \langle Y_{\delta,j}, \pi^-_{-\bar\lambda}(x)Y_0\rangle$, i.e. $\Phi^j_{\lambda,\delta}$ is a matrix coefficient of the spherical principal series in the compact picture (see §4). It is well-known (see [17]) that they are eigenfunctions of the Laplace–Beltrami operator Δ with eigenvalues $-(\lambda^2+\rho_0^2)$. When δ is trivial then $\Phi^1_{\lambda,\delta}$ is obviously the elementary spherical function ϕ_λ.

The following result can be viewed as an analogue of (3.6). For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $x = ka_rK \in X$ and $1 \le j \le d_\delta$ (see [17], p. 344)

$$\Phi^j_{\lambda,\delta}(x) = Y_{\delta,j}(kM)\Phi^1_{\lambda,\delta}(a_r). \tag{5.3}$$

Elements $\delta \in \widehat{K}_0$ can be parametrized by a pair of integers (p_δ, q_δ) so that $p_\delta \ge 0$ and $p_\delta \pm q_\delta \in 2\mathbb{Z}^+$ (see [20,22]). The trivial representation in $\widehat{K}_0$ is parametrized by $(0,0)$ in this set-up. Every $m > 0$ determines a subset $\widehat{K}_0(m)$ of $\widehat{K}_0$ by $\widehat{K}_0(m) = \{\delta \in \widehat{K}_0\colon p_\delta < m\}$. This set is finite because, $p_\delta \ge |q_\delta|$. This parametrization of $\widehat{K}_0$ will make a crucial appearance in our results. We shall come back to that, after a digression.

Jacobi functions. At this point we give a quick review of some preliminaries on the Jacobi functions. For a detailed exposition the reader is referred to [21].

For $\alpha, \beta, \lambda \in \mathbb{C}, \alpha$ not a negative integer and $r \in \mathbb{R}$, let $\phi^{(\alpha,\beta)}_\lambda(r)$ be the Jacobi function of type (α,β) which is given in terms of the hypergeometric function ${}_2F_1$ as

$$\phi^{(\alpha,\beta)}_\lambda(r) = {}_2F_1\left(\frac{\alpha+\beta+1+i\lambda}{2}, \frac{\alpha+\beta+1-i\lambda}{2}; \alpha+1; -(\sinh r)^2\right).$$

Let

$$\mathscr{L}_{\alpha,\beta} = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + ((2\alpha+1)\coth r + (2\beta+1)\tanh r)\frac{\mathrm{d}}{\mathrm{d}r}$$

be the Jacobi Laplacian and let $\rho = \alpha+\beta+1$. Then $\phi^{(\alpha,\beta)}_\lambda(r)$ is the unique analytic solution of the equation $\mathscr{L}_{\alpha,\beta}\phi = -(\lambda^2+\rho^2)\phi$, which is even and $\phi^{(\alpha,\beta)}_\lambda(0) = 1$. They also satisfy respectively the following relation and the estimate: For $r \in \mathbb{R}^+$ and $\lambda \in \mathbb{C}$,

$$\phi^{\alpha,\beta}_\lambda(r) = (\cosh r)^{-2\beta}\phi^{\alpha,-\beta}_\lambda(r) \tag{5.4}$$

and

$$|\phi^{(\alpha,\beta)}_\lambda(r)| \le C(1+r)\mathrm{e}^{r(|\lambda_I|-\rho)}. \tag{5.5}$$

The associated Jacobi function $\phi^{(\alpha,\beta)}_{\lambda,p,q}$ for two extra parameters $p, q \in \mathbb{Z}$ is defined as

$$\phi^{(\alpha,\beta)}_{\lambda,p,q}(r) = (\sinh r)^p(\cosh r)^q\phi^{(\alpha+p,\beta+q)}_\lambda(r).$$

We also have the associated Jacobi operator,

$$\begin{aligned}\mathscr{L}_{\alpha,\beta,p,q} = {}& \frac{\mathrm{d}^2}{\mathrm{d}r^2} + ((2\alpha+1)\coth r + (2\beta+1)\tanh r)\frac{\mathrm{d}}{\mathrm{d}r} \\ & + \{-(2\alpha+p)p(\sinh r)^{-2} + (2\beta+q)q(\cosh r)^{-2}\}.\end{aligned}$$

One can easily verify that $\phi^{(\alpha,\beta)}_{\lambda,p,q}$ is again the unique solution of $\mathscr{L}_{\alpha,\beta,p,q}\phi = -(\lambda^2+\rho^2)\phi$, which is even and satisfies $\phi^{(\alpha,\beta)}_{\lambda,p,q}(0) = 1$, in the case $p = 0$. The proof of our next theorem will involve finding a relation between the heat kernels of the operators $\mathscr{L}_{\alpha+p,\beta+q}$ and $\mathscr{L}_{\alpha,\beta,p,q}$.

Let

$$\Delta_{\alpha,\beta}(r) = (2\sinh r)^{2\alpha+1}(2\cosh r)^{2\beta+1} = 4^{\rho}(\sinh r)^{2\alpha+1}(\cosh r)^{2\beta+1}. \quad (5.6)$$

Then the (Fourier-) Jacobi transform of a suitable function f on $\mathbb{R}^+$ is defined by

$$J_{\alpha,\beta}(f)(\lambda) = \int_0^\infty f(r)\phi^{(\alpha,\beta)}_{\lambda}(r)\Delta_{\alpha,\beta}(r)\mathrm{d}r, \lambda \in \mathbb{C}. \quad (5.7)$$

The *heat kernel* $h^{(\alpha,\beta)}_t$, associated to $\mathscr{L}_{\alpha,\beta}$ is defined by

$$\int_0^\infty h^{(\alpha,\beta)}_t(r)\phi^{(\alpha,\beta)}_{\lambda}(r)\Delta_{\alpha,\beta}(r)\mathrm{d}r = \mathrm{e}^{-(\lambda^2+\rho^2)t}. \quad (5.8)$$

If $\alpha > -1$ and $\alpha \pm \beta \geq -1$, then we have *inversion formula* for the Jacobi transform [21] from which we get

$$h^{(\alpha,\beta)}_t(r) = \int_0^\infty \mathrm{e}^{-(\lambda^2+\rho^2)t}\phi^{(\alpha,\beta)}_{\lambda}(r)|c_{\alpha,\beta}(\lambda)|^{-2}\mathrm{d}\lambda, \quad (5.9)$$

where

$$c_{\alpha,\beta}(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+\rho))\Gamma(\frac{1}{2}(i\lambda+\alpha-\beta+1))}.$$

We will see below that the type of α,β we are interested in satisfy the above restriction of α,β and hence the inversion formula is valid there.

We have the following sharp estimate of $h^{(\alpha,\beta)}_t$, due to Anker *et al* [4].

Theorem 5.1. *Let $\alpha,\beta \in \mathbb{Z}$ and $\alpha \geq \beta \geq -\frac{1}{2}$. Then for $t > 0$,*

$$h^{(\alpha,\beta)}_t(r) \asymp t^{-3/2}\mathrm{e}^{-\rho^2 t}(1+r)\left(1+\frac{1+r}{t}\right)^{\alpha-\frac{1}{2}}\mathrm{e}^{-\rho r}\mathrm{e}^{-r^2/4t}. \quad (5.10)$$

Back to symmetric space. Let us now come back to the symmetric space X and assume that $\alpha = \frac{m_\gamma+m_{2\gamma}-1}{2}$ and $\beta = \frac{m_{2\gamma}-1}{2}$. Then $\rho = \alpha+\beta+1$ is the same as $\rho_0 = \frac{m_\gamma+2m_{2\gamma}}{2}$. In this case the operator $\mathscr{L}_{\alpha,\beta}$ coincides with the radial part of the Laplace–Beltrami operator Δ of X. In the previous section we have noticed that the usual heat kernel is bi-invariant. Therefore, $\Delta h_t = \mathscr{L}_{\alpha,\beta}h_t$ and consequently, the heat kernel of $\mathscr{L}_{\alpha,\beta}$, $h^{(\alpha,\beta)}_t(r)$ is the same as the heat kernel $h_t(a_r)$ of Δ. Note also that in this case, $\mu(\lambda) = |c(\lambda)|^{-2} = |c_{\alpha,\beta}(\lambda)|^{-2} \asymp (1+|\lambda|)^{2\alpha+1}$ by (4.4).

In all rank 1 symmetric spaces except the real hyperbolic space, α and β satisfy the restriction in Theorem 5.1 and hence have the estimate (5.10). (A full chart of α,β etc. for the rank 1 symmetric spaces is available for instance in [11].) On the other hand for

real hyperbolic space $\mathscr{H}^n = SO(n,1)/SO(n)$, where $\alpha = \frac{n-2}{2}$ and $\beta = -\frac{1}{2}$, Davies and Mandouvalos [9] already has the estimate of the type (5.10) for the heat kernel h_t.

For $\delta \in \widehat{K}_0, h_t^\delta = h_t^{(\alpha+p_\delta,\beta+q_\delta)}$ denotes the heat kernel corresponding to the operator $\mathscr{L}_{\alpha+p_\delta,\beta+q_\delta}$. By definition of the heat kernel $J_\delta(h_t^\delta) = J_{\alpha+p_\delta,\beta+q_\delta}(h_t^\delta) = \mathrm{e}^{-(\lambda^2+\rho_\delta^2)t}$, where $\rho_\delta = \alpha + p_\delta + \beta + q_\delta + 1$. Notice that $\alpha + p_\delta, \beta + q_\delta$ are just two other α, β. When both p_δ, q_δ are non-negative integers and α, β satisfies the hypothesis of Theorem 5.1, $\alpha + p_\delta, \beta + q_\delta$ will also satisfy the hypothesis and consequently h_t^δ will again have the same estimate with ρ_δ replacing ρ. There are however two exceptional cases.

(A) As noted earlier in the real hyperbolic spaces $\mathscr{H}^n$, α, β do not satisfy the restriction in Theorem 5.1 and from [9] we get the estimate (5.10) only for the bi-invariant heat kernel h_t. To get a similar estimate for h_t^δ, where δ is not the trivial representation, we point out that in $SO(n,1), q_\delta = 0$ for all δ and that for (α,β) of $\mathscr{H}^n$, $(\alpha + p_\delta, \beta)$ are the corresponding parameters for the higher dimensional hyperbolic space $\mathscr{H}^{n+2p_\delta}$ (see [11]). That is, h_t^δ of $\mathscr{H}^n$ is the same as h_t of $\mathscr{H}^{n+2p_\delta}$. Thus we can again appeal to the result of [9] to show that h_t^δ satisfies the estimate (5.10) substituting α by $\alpha + p_\delta$.

(B) In the complex hyperbolic space $SU(n,1)/S(U(n)\times U(1)), p_\delta \geq 0, p_\delta > |q_\delta|$, but q_δ can be negative. (In all other rank 1 symmetric spaces both p_δ and q_δ are non-negative.) To handle this exception, we shall denote by $\tilde{\delta}$ the representation in $\widehat{K}_0$ which corresponds to $(p_\delta, |q_\delta|)$. That is $|q_\delta| = q_{\tilde{\delta}}$. To have a unform approach, in all symmetric spaces, instead of (p_δ, q_δ), we will consider $(p_\delta, |q_\delta|)$ and deal with $h_t^{\tilde{\delta}}$, which will clearly satisfy (5.10). Note that except for $SU(n,1)/S(U(n)\times U(1)), h_t^{\tilde{\delta}}$ is merely h_t^δ as $|q_\delta| = q_\delta$.

It is also clear that all α, β as well as $\alpha + p_\delta, \beta + |q_\delta|$ we are concerned with satisfy the restriction for the validity of the inversion formula (5.9) (see [11] p. 265 for α,β's and [20,22] for p_δ, q_δ's).

We proceed to find the heat kernel of the operator $\mathscr{L}_{\alpha,\beta,p_\delta,q_\delta}$. The following expansion relates the generalized spherical functions with the Jacobi functions:

$$\begin{aligned}\Phi^1_{\lambda,\delta}(a_r) &= Q_\delta(\lambda)(\alpha+1)^{-1}_{p_\delta}(\sinh r)^{p_\delta}(\cosh r)^{q_\delta}\phi_\lambda^{\alpha+p_\delta,\beta+q_\delta}(r)\\ &= Q_\delta(\lambda)(\alpha+1)^{-1}_{p_\delta}\phi^{\alpha,\beta}_{\lambda,p_\delta,q_\delta}(r),\end{aligned} \tag{5.11}$$

where $x = ka_rK, Q_\delta(\lambda) = (\frac{1}{2}(\alpha+\beta+1+i\lambda))_{\frac{p_\delta+q_\delta}{2}}(\frac{1}{2}(\alpha-\beta+1+i\lambda))_{\frac{p_\delta-q_\delta}{2}})$ is the Kostant polynomial and $(z)_m = \Gamma(z+m)/\Gamma(z)$. This relation is due to Helgason (see [18]). We have used the parametrization given in [5]. It follows in particular from (5.11) that $\phi_\lambda^{(\alpha,\beta)}(r)$ coincides with the elementary spherical function $\phi_\lambda(a_r)$ and that $\Phi^1_{\lambda,\delta}$ is an eigenfunction of $\mathscr{L}_{\alpha,\beta,p_\delta,q_\delta}$ with the eigenvalue $-(\lambda^2+\rho_0^2)$.

For a function f on X with a suitable decay, the jth component of the *δ-spherical transform* of f is

$$\widehat{f}_{\delta,j}(\lambda) = \int_X f(x)\Phi^j_{-\lambda,\delta}(x)\mathrm{d}x$$

(see [5,17]). It is easy to see that $\widehat{f}_{\delta,j}(\overline{\lambda}) = \widehat{f}_u(\lambda)$ defined in the previous section when $u = Y_{\delta,j}$.

We define a function $H^\delta(x,t) = H_t^\delta(x)$ on $X \times R^+$, through its δ-spherical transform as $\widehat{H^\delta}_{t\,\delta,j}(\lambda) = Q_\delta(\lambda)\mathrm{e}^{-(\lambda^2+\rho_0^2)t}$, for $1 \leq j \leq d_\delta$ and $\widehat{H^\delta}_{t\,\delta',j'} \equiv 0$ for $1 \leq j' \leq d_{\delta'}$, when $\delta' \neq \delta$. Clearly for each fixed $t > 0$, H_t^δ is a function on X of left K-type δ. Since $\Phi^j_{\lambda,\delta}$ is an eigenfunction of Δ with eigenvalue $-(\lambda^2+\rho_0^2)$, it is easy to see that H_t^δ is a solution of type δ of the heat equation, $\Delta f = \frac{\partial}{\partial t} f$. Note that the Fourier transform of H_t^δ has only the generic zeroes of the Fourier transforms of functions of left-type δ. In that way it is the basic solution (of type δ) of the heat equation and may be viewed as a generalization of the bi-invariant heat kernel to arbitrary K-types. We shall see that $H_t^\delta(x)$ also satisfies an estimate similar to that in Theorem 5.1.

Let us recall that $Y_{\delta,j}$'s ($\delta \in \widehat{K}_0, 1 \leq j \leq d_\delta$) are spherical harmonics. In analogy with (3.3), for a nice measurable function f on X we can define, $F_{\delta,j}(f)(\lambda) = Q_\delta(-\lambda)^{-1}\int_{K/M} \tilde{f}(\lambda, kM) Y_{\delta,j}(kM)\mathrm{d}k$ and $\tilde{F}_{\delta,j}(f)(\lambda) = Q_\delta(-\lambda)F_{\delta,j}(f)(\lambda)$.

Remark 5.2. Looking back at the results of §4 we see that explicit knowledge about the matrix coefficients of the principal series representations may lead to a refinement of the characterization given in Theorems 4.1 and 4.3. For instance here when G is of real rank 1, if $\deg Q_\delta \geq P_{u,b}$ where u transform according to $\delta \in \widehat{K}$ and Q_δ is the Kostant polynomial defined above then $f_{u,0} = 0$. Therefore if $\deg Q_\delta > \frac{2}{p'} + \frac{k}{p}$ or $\deg Q_\delta \geq \frac{l-2\alpha-1}{q}$, then $f_\delta = 0$. Unfortunately no exact description of the properties of the matrix coefficients of the representations is available in general rank, because such a description needs an exhaustive understanding of the subquotients of the principal series representations.

With this preparation we are now in a position to state our last result:

Theorem 5.3. *Let f be a measurable function on X and $p,q \in [1,\infty)$. Suppose for some $k,l \in \mathbb{R}^+$,*

$$\int_X \frac{|f(x)h_t(x)^{-1}\Xi(x)^{\frac{2}{p}}|^p}{(1+\sigma(x))^k}\mathrm{d}x < \infty \tag{5.12}$$

and for every $\delta \in \widehat{K}_0$ and $0 \leq j \leq d_\delta$,

$$\int_{\mathfrak{a}^*} \frac{|F_{\delta,j}(f)(\lambda)\mathrm{e}^{t|\lambda|^2}|^q}{(1+|\lambda|)^l}|c(\lambda)|^{-2}\mathrm{d}\lambda < \infty. \tag{5.13}$$

If also $l \leq q+2\alpha+2$, then $f = \Sigma_{\delta\in\widehat{K}_0(\frac{k-1}{p})} f_\delta$ is left K-finite where f_δ is the left δ-isotypic component of f and

(a) *if for $\delta \in \widehat{K}_0(\frac{k-1}{p})$ and $1 \leq j \leq d_\delta, F_{\delta,j}(f)(i\rho_0) = 1$ or $F_{\delta,j}(f)(-i\rho_0) = 1$ according as $Q_\delta(i\rho_0) \neq 0$ or $Q_\delta(-i\rho_0) \neq 0$, then*

$$f(a_r) = \Sigma_{\delta\in\widehat{K}_0(\frac{k-1}{p})} H_t^\delta(a_r).$$

(b) *if $k \leq p+1$ and $\int_X f(x)\mathrm{d}x = 1$, then $f = H_t^0 = h_t$.*

Proof. Let $f_{\delta,j}(a_r) = \int_K f(ka_r)Y_{\delta,j}(kM)\mathrm{d}k$. Then $f_{\delta,j}$ satisfies (5.12).

From the definition of the Helgason Fourier transform (5.1) we have

$$F_{\delta,j}(f)(\lambda) = Q_\delta(\lambda)^{-1} \int_X \int_{K/M} f(x) \mathrm{e}^{(-i\lambda+\rho)A(x,b)} Y_{\delta,j}(b) \mathrm{d}b \, \mathrm{d}x.$$

Let $x = ka_rK$. Using (5.3) and then changing over to polar coordinates we get,

$$\begin{aligned} F_{\delta,j}(f)(\lambda) &= Q_\delta(\lambda)^{-1} \int_X f(ka_r) Y_{\delta,j}(kM) \Phi^1_{-\lambda,\delta}(a_r) \mathrm{d}x \\ &= Q_\delta(\lambda)^{-1} \int_0^\infty f_{\delta,j}(a_r) \Phi^1_{-\lambda,\delta}(a_r) \Delta_{\alpha,\beta}(r) \mathrm{d}r. \end{aligned}$$

Using (5.11) and (5.6), we further have,

$$F_{\delta,j}(f)(\lambda) = \frac{4^{p_\delta+q_\delta}}{(\alpha+\beta)_{p_\delta}} \int_0^\infty f^\delta_{\delta,j}(a_r) \phi^{(\alpha+p_\delta,\beta+q_\delta)}_{-\lambda}(r) \Delta_{\alpha+p_\delta,\beta+q_\delta}(r) \mathrm{d}r, \tag{5.14}$$

where

$$f^{\delta'}_{\delta,j}(a_r) = f_{\delta,j}(a_r)(\sinh r)^{-p_{\delta'}}(\cosh r)^{-q_{\delta'}}. \tag{5.15}$$

Note that in the lone case (namely in $G = SU(n,1)$) where q_δ can be negative and hence $|q_\delta| \neq q_\delta, \beta$ is zero. Therefore in view of the discussion (B) above we will rewrite (5.14) using (5.4) as

$$F_{\delta,j}(f)(\lambda) = C_\delta \int_0^\infty f^{\tilde{\delta}}_{\delta,j}(a_r) \phi^{(\alpha+p_\delta,\beta+|q_\delta|)}_{-\lambda}(r) \Delta_{\alpha+p_\delta,\beta+|q_\delta|}(r) \mathrm{d}r. \tag{5.16}$$

We can now follow exactly the same steps as of Theorem 4.1 and use (5.12), (5.5), (5.6) and finally appeal to Lemma 2.2 to show that $F_{\delta,j}(f)(\lambda) = C^t_{\delta,j} \cdot \mathrm{e}^{-t\lambda^2}$.

But by (5.16), $F_{\delta,j}(f)(\lambda)$ is the Jacobi transform of type $(\alpha+p_\delta, \beta+|q_\delta|)$ of $f^{\tilde{\delta}}_{\delta,j}$. Therefore from (5.9) and (5.15),

$$f_{\delta,j}(a_r) = C_{\delta,j}(\sinh r)^{p_\delta}(\cosh r)^{|q_\delta|} h^{\tilde{\delta}}_t(r). \tag{5.17}$$

On the other hand from (5.1) and (5.2) we have

$$\begin{aligned} \tilde{F}_{\delta,j}(f)(\lambda) &= \int_{K/M} \tilde{f}(\lambda, kM) Y_{\delta,j}(kM) \mathrm{d}k \\ &= \int_X \int_{K/M} f(x) \mathrm{e}^{(-i\lambda+\rho)A(x,b)} Y_{\delta,j}(b) \mathrm{d}b \, \mathrm{d}x \\ &= \int_X f(x) \Phi^j_{-\lambda,\delta}(x) \mathrm{d}x \\ &= \widehat{f}_{\delta,j}(\lambda). \end{aligned}$$

Therefore, $F_{\delta,j}(f)(\lambda) = C_{\delta,j} \cdot \mathrm{e}^{-t\lambda^2}$ for $\lambda \in \mathbb{R}$ implies that $f_{\delta,j}(x) = C^t_{\delta,j} H^\delta_{t,j}(x)$. That is,

$$f_{\delta,j}(a_r) = C^t_{\delta,j}(\sinh r)^{p_\delta}(\cosh r)^{|q_\delta|} h^{\tilde{\delta}}_t(r) = C^t_{\delta,j} H^\delta_{t,j}(a_r). \tag{5.18}$$

The apparent contradiction that arises on taking the Fourier transform of the sides can be resolved by noting that $Q_\delta = Q_{\tilde{\delta}}$ because when $|q_\delta| \neq q_\delta$ then $\beta = 0$.

Now from Theorem 5.1 and the subsequent discussions (A) and (B), it follows that

$$\begin{aligned} H_t^\delta(a_r) &\asymp C(\delta,t)(1+r)\left(1+\frac{1+r}{t}\right)^{\alpha+p_\delta-\frac{1}{2}} \mathrm{e}^{-(\rho_0 r+\frac{r^2}{4t})} \\ &\asymp C(\delta,t)\left(1+\frac{1+r}{t}\right)^{p_\delta} h_t(r). \end{aligned} \tag{5.19}$$

Notice that we have switched from q_δ to $|q_\delta|$ to fulfill the requirement of Theorem 5.1.

If $\delta \in \widehat{K}_0 \setminus \widehat{K}_0(\frac{k-1}{p})$, then $p_\delta \geq \frac{k-1}{p}$ and hence $f_{\delta,j} \equiv 0$ for $j = 1,\ldots,d_\delta$ by (5.12) and (5.19).

Now (a) follows by taking Fourier transforms of the two sides of $f_{\delta,j}(a_r) = C^t_{\delta,j} H_t^\delta(a_r)$ and putting the initial condition.

If $k \leq p+1$, then $p_\delta \geq \frac{k-1}{p}$ whenever δ is non-trivial as $p_\delta > |q_\delta|$. Therefore, $f = C_t h_t$. In particular f is a bi-invariant function. Note that $\int_X f(x)\mathrm{d}x = \widehat{f}(i\rho_0)_{0,0}$, where $\widehat{f}(\cdot)_{0,0}$ is the spherical Fourier transform of f. Using the condition $\int_X f(x)\mathrm{d}x = \widehat{f}(i\rho_0)_{0,0} = 1$, we get $f = h_t$. Thus (b) is proved.

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